Day 1 Logs

A driver with a concentration of alcohol in the blood of 0.094 or more should be arrested and charged with DUI. Verify using a graphing utility.

NOTE: Most states use 0.08 or 0.10 as the blood alcohol content at which a DUI citation is given.

SUMMARY

Properties of the Logarithmic Function

$f(x) = \log_a x, \quad a > 1$ ($y = \log_a x$ means $x = a^y$)
Domain: $(0, \infty)$; Range: $(-\infty, \infty)$; x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); increasing; one-to-one

See Figure 26(b) for a typical graph.

$f(x) = \log_a x, \quad 0 < a < 1$ ($y = \log_a x$ means $x = a^y$)
Domain: $(0, \infty)$; Range: $(-\infty, \infty)$; x-intercept: 1; y-intercept: none; vertical asymptote: $x = 0$ (y-axis); decreasing; one-to-one

See Figure 26(a) for a typical graph.

5.3 EXERCISES

In Problems 1–12, change each exponential expression to an equivalent expression involving a logarithm.

1. $9 = 3^2$
2. $16 = 4^2$
3. $a^2 = 1.6$
4. $a^3 = 2.1$
5. $1.1^2 = M$
6. $2.2^3 = N$
7. $2^x = 7.2$
8. $3^x = 4.6$
9. $x^{\sqrt{2}} = \pi$
10. $x^{\pi} = e$
11. $e^x = 8$
12. $e^{2.2} = M$

In Problems 13–24, change each logarithmic expression to an equivalent expression involving an exponent.

13. $\log_2 8 = 3$
14. $\log_3\left(\frac{1}{9}\right) = -2$
15. $\log_a 3 = 6$
16. $\log_b 4 = 2$
17. $\log_3 2 = x$
18. $\log_2 6 = x$
19. $\log_2 M = 1.3$
20. $\log_3 N = 2.1$
21. $\log_{\sqrt{2}} \pi = x$
22. $\log_{\pi} x = \frac{1}{2}$
23. $\ln 4 = x$
24. $\ln x = 4$

In Problems 25–36, find the exact value of each logarithm without using a calculator.

25. $\log_2 1$
26. $\log_8 8$
27. $\log_5 25$
28. $\log_3\left(\frac{1}{9}\right)$
29. $\log_{1/2} 16$
30. $\log_{1/3} 9$
31. $\log_{10} \sqrt{10}$
32. $\log_5 \sqrt[3]{25}$
33. $\log_{\sqrt{2}} 4$
34. $\log_{\sqrt{3}} 9$
35. $\ln \sqrt{e}$
36. $\ln e^3$

In Problems 37–46, find the domain of each function.

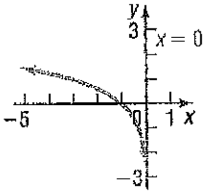
37. $f(x) = \ln(x - 3)$
38. $g(x) = \ln(x - 1)$
39. $F(x) = \log_2 x^2$
40. $H(x) = \log_5 x^3$
41. $h(x) = \log_{1/2}(x^2 - 2x + 1)$
42. $G(x) = \log_{1/2}(x^2 - 1)$
43. $f(x) = \ln\left(\frac{1}{x + 1}\right)$
44. $g(x) = \ln\left(\frac{1}{x - 5}\right)$
45. $g(x) = \log_5\left(\frac{x + 1}{x}\right)$
46. $h(x) = \log_3\left(\frac{x}{x - 1}\right)$

In Problems 47–50, use a calculator to evaluate each expression. Round your answer to three decimal places.

47. $\ln\frac{5}{3}$
48. $\frac{\ln 5}{3}$
49. $\frac{\ln(10/3)}{0.04}$
50. $\frac{\ln(2/3)}{-0.1}$
51. Find a so that the graph of $f(x) = \log_a x$ contains the point $(2, 2)$.
52. Find a so that the graph of $f(x) = \log_a x$ contains the point $\left(\frac{1}{2}, -4\right)$.

In Problems 53–60, the graph of a logarithmic function is given. Match each graph to one of the following functions:

A. $y = \log_3 x$	*B.* $y = \log_3(-x)$	*C.* $y = -\log_3 x$	*D.* $y = -\log_3(-x)$
E. $y = \log_3 x - 1$	*F.* $y = \log_3(x - 1)$	*G.* $y = \log_3(1 - x)$	*H.* $y = 1 - \log_3 x$

53.


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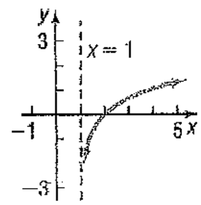

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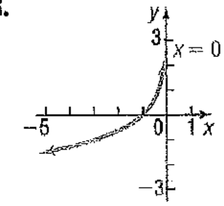

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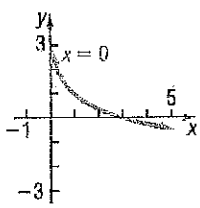

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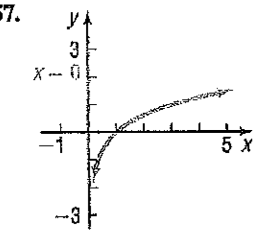

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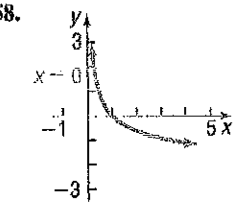

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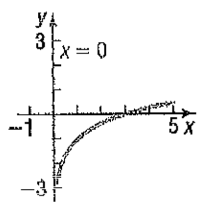

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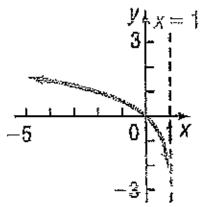

In Problems 61–72, use transformations to graph each function. Determine the domain, range, and vertical asymptote of each function. Verify your results using a graphing utility.

61. $f(x) = \ln(x + 4)$ **62.** $f(x) = \ln(x - 3)$ **63.** $f(x) = \ln(-x)$ **64.** $f(x) = -\ln(-x)$
65. $g(x) = \ln(2x)$ **66.** $h(x) = \ln\left(\frac{1}{2}x\right)$ **67.** $f(x) = 3\ln x$ **68.** $f(x) = -2\ln x$
69. $g(x) = \ln(3 - x)$ **70.** $h(x) = \ln(4 - x)$ **71.** $f(x) = -\ln(x - 1)$ **72.** $f(x) = 2 - \ln x$

In Problems 73–88, solve each equation.

73. $\log_3 x = 2$ **74.** $\log_5 x = 3$ **75.** $\log_2(2x + 1) = 3$ **76.** $\log_3(3x - 2) = 2$
77. $\log_x 4 = 2$ **78.** $\log_x\left(\frac{1}{8}\right) = 3$ **79.** $\ln e^x = 5$ **80.** $\ln e^{-2x} = 8$
81. $\log_4 64 = x$ **82.** $\log_5 625 = x$ **83.** $\log_3 243 = 2x + 1$ **84.** $\log_6 36 = 5x + 3$
85. $e^{3x} = 10$ **86.** $e^{-2x} = \frac{1}{3}$ **87.** $e^{2x+5} = 8$ **88.** $e^{-2x+1} = 13$
89. $\log_2(x^2 + 1) = 2$ **90.** $\log_5(x^2 + x + 4) = 2$ **91.** $\log_2 8^x = -3$ **92.** $\log_3 3^x = 1$

In Problems 93–98, (a) graph each function and state its domain, range, and asymptote; (b) determine the inverse function; (c) use the graph obtained in (a) to graph the inverse and state its domain, range, and asymptote.

93. $f(x) = 2^x$ **94.** $f(x) = 5^x$ **95.** $f(x) = 2^{x+3}$ **96.** $f(x) = 5^x - 2$
97. $f(x) = 1 + 2^{x-3}$ **98.** $f(x) = 5^{x+1} - 2$

SUMMARY

Properties of Logarithms

In the list that follows, $a > 0, a \neq 1$, and $b > 0, b \neq 1$; also, $M > 0$ and $N > 0$.

Definition	$y = \log_a x$ means $x = a^y$
Properties of logarithms	$\log_a 1 = 0;\quad \log_a a = 1$
	$a^{\log_a M} = M;\quad \log_a a^r = r$
	$\log_a(MN) = \log_a M + \log_a N$
	$\log_a\left(\dfrac{M}{N}\right) = \log_a M - \log_a N$
	$\log_a\left(\dfrac{1}{N}\right) = -\log_a N$
	$\log_a M^r = r\log_a M$
Change-of-Base Formula	$\log_a M = \dfrac{\log_b M}{\log_b a}$

HISTORICAL FEATURE

John Napier (1550–1617)

Logarithms were invented about 1590 by John Napier (1550–1617) and Joost Bürgi (1552–1632), working independently. Napier, whose work had the greater influence, was a Scottish lord, a secretive man whose neighbors were inclined to believe him to be in league with the devil. His approach to logarithms was very different from ours; it was based on the relationship between arithmetic and geometric sequences (see Chapter 12 on induction and sequences) and not on the inverse function relationship of logarithms to exponential functions (described in Section 5.3). Napier's tables, published in 1614, listed what would now be called *natural logarithms* of sines and were rather difficult to use. A London professor, Henry Briggs, became interested in the tables and visited Napier. In their conversations, they developed the idea of common logarithms, which were published in 1617. Their importance for calculation was immediately recognized, and by 1650 they were being printed as far away as China. They remained an important calculation tool until the advent of the inexpensive handheld calculator about 1972, which has decreased their calculational, but not their theoretical, importance.

A side effect of the invention of logarithms was the popularization of the decimal system of notation for real numbers.

5.4 EXERCISES

In Problems 1–12, suppose that $\ln 2 = a$ *and* $\ln 3 = b$. *Use properties of logarithms to write each logarithm in terms of a and b.*

1. $\ln 6$ **2.** $\ln\frac{2}{3}$ **3.** $\ln 1.5$ **4.** $\ln 0.5$

5. $\ln(2e)$ **6.** $\ln\left(\dfrac{3}{e}\right)$ **7.** $\ln 12$ **8.** $\ln 24$

9. $\ln\sqrt[5]{18}$ **10.** $\ln\sqrt[4]{48}$ **11.** $\log_2 3$ **12.** $\log_3 2$

In Problems 13–28, write each expression as a sum and/or difference of logarithms. Express powers as factors.

13. $\log_a(u^2v^3)$ **14.** $\log_2\left(\dfrac{a}{b^2}\right)$ **15.** $\log\dfrac{1}{M^3}$ **16.** $\log(10u^2)$

17. $\log_5\sqrt{\dfrac{a^3}{b}}$ **18.** $\log_6\left(\dfrac{ab^4}{\sqrt[3]{c^2}}\right)$ **19.** $\ln(x^2\sqrt{1-x})$ **20.** $\ln(x\sqrt{1+x^2})$

21. $\log_2\left(\dfrac{x^3}{x-3}\right)$ **22.** $\log_5\left(\dfrac{\sqrt[3]{x^2+1}}{x^2-1}\right)$ **23.** $\log\left[\dfrac{x(x+2)}{(x+3)^2}\right]$ **24.** $\log\left[\dfrac{x^3\sqrt{x+1}}{(x-2)^2}\right]$

25. $\ln\left[\frac{x^2-x-2}{(x+4)^2}\right]^{1/3}$ **26.** $\ln\left[\frac{(x-4)^2}{x^2-1}\right]^{2/3}$ **27.** $\ln\frac{5x\sqrt{1-3x}}{(x-4)^3}$ **28.** $\ln\left[\frac{5x^2\sqrt[3]{1-x}}{4(x+1)^2}\right]$

In Problems 29–38, write each expression as a single logarithm.

29. $3\log_5 u + 4\log_5 v$

30. $\log_3 u^2 - \log_3 v$

31. $\log_{1/2}\sqrt{x} - \log_{1/2}x^3$

32. $\log_2\left(\frac{1}{x}\right) + \log_2\left(\frac{1}{x^2}\right)$

33. $\ln\left(\frac{x}{x-1}\right) + \ln\left(\frac{x+1}{x}\right) - \ln(x^2-1)$

34. $\log\left(\frac{x^2+2x-3}{x^2-4}\right) - \log\left(\frac{x^2+7x+6}{x+2}\right)$

35. $8\log_2\sqrt{3x-2} - \log_2\left(\frac{4}{x}\right) + \log_2 4$

36. $21\log_3\sqrt[3]{x} + \log_3(9x^2) - \log_3 25$

37. $2\log_a(5x^3) - \frac{1}{2}\log_a(2x+3)$

38. $\frac{1}{3}\log(x^3+1) + \frac{1}{2}\log(x^2+1)$

39. Write the exponential model $y = ab^x$ as a linear model. [**Hint:** Take the logarithm of both sides.]

40. Write the power model $y = ax^b$ as a linear model.

In Problems 41–44, find the exact value of each expression.

41. $3^{\log_3 5 - \log_3 4}$ **42.** $5^{\log_5 6 + \log_5 7}$ **43.** $e^{\log_{e^2} 16}$ **44.** $e^{\log_{e^3} 9}$

In Problems 45–52, use the Change-of-Base Formula and a calculator to evaluate each logarithm. Round your answer to three decimal places.

45. $\log_3 21$ **46.** $\log_5 18$ **47.** $\log_{1/3} 71$ **48.** $\log_{1/2} 15$

49. $\log_{\sqrt{2}} 7$ **50.** $\log_{\sqrt{5}} 8$ **51.** $\log_\pi e$ **52.** $\log_\pi \sqrt{2}$

In Problems 53–58, graph each function using a graphing utility and the Change-of-Base Formula.

53. $y = \log_4 x$ **54.** $y = \log_5 x$ **55.** $y = \log_2(x+2)$ **56.** $y = \log_4(x-3)$

57. $y = \log_{x-1}(x+1)$ **58.** $y = \log_{x+2}(x-2)$

In Problems 59–68, express y as a function of x. The constant C is a positive number.

59. $\ln y = \ln x + \ln C$

60. $\ln y = \ln(x + C)$

61. $\ln y = \ln x + \ln(x+1) + \ln C$

62. $\ln y = 2\ln x - \ln(x+1) + \ln C$

63. $\ln y = 3x + \ln C$

64. $\ln y = -2x + \ln C$

65. $\ln(y-3) = -4x + \ln C$

66. $\ln(y+4) = 5x + \ln C$

67. $3\ln y = \frac{1}{2}\ln(2x+1) - \frac{1}{3}\ln(x+4) + \ln C$

68. $2\ln y = -\frac{1}{2}\ln x + \frac{1}{3}\ln(x^2+1) + \ln C$

69. Find the value of $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$.

70. Find the value of $\log_2 4 \cdot \log_4 6 \cdot \log_6 8$.

71. Find the value of $\log_2 3 \cdot \log_3 4 \cdot \ldots \cdot \log_n(n+1) \cdot \log_{n+1} 2$.

72. Find the value of $\log_2 2 \cdot \log_2 4 \cdot \ldots \cdot \log_2 2^n$.

73. Show that $\log_a(x + \sqrt{x^2-1}) + \log_a(x - \sqrt{x^2-1}) = 0$.

74. Show that $\log_a(\sqrt{x} + \sqrt{x-1}) + \log_a(\sqrt{x} - \sqrt{x-1}) = 0$.

75. Show that $\ln(1 + e^{2x}) = 2x + \ln(1 + e^{-2x})$.

76. If $f(x) = \log_a x$, show that $\frac{f(x+h) - f(x)}{h} = \log_a\left(1 + \frac{h}{x}\right)^{1/h}$, $h \neq 0$.

77. If $f(x) = \log_a x$, show that $-f(x) = \log_{1/a} x$.

78. If $f(x) = \log_a x$, show that $f(AB) = f(A) + f(B)$.

79. If $f(x) = \log_a x$, show that $f(1/x) = -f(x)$.

80. If $f(x) = \log_a x$, show that $f(x^\alpha) = \alpha f(x)$.

81. Show that $\log_a(M/N) = \log_a M - \log_a N$, where a, M, and N are positive real numbers, with $a \neq 1$.

82. Show that $\log_a(1/N) = -\log_a N$, where a and N are positive real numbers, with $a \neq 1$.

83. Find the domain of $f(x) = \log_a x^2$ and the domain of $g(x) = 2\log_a x$. Since $\log_a x^2 = 2\log_a x$, how do you reconcile the fact that the domains are not equal? Write a brief explanation.